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Complex Geometry and Supersymmetry

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I stress how the form of sigma models with $(2,2)$ supersymmetry differs depending on the number of manifest supersymmetries. The differences correspond to different aspects/formulations of Generalized Kähler Geometry.

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1. Introduction

In this brief presentation I report on an aspects of the relation between twodimensional $N=(2,2)$ sigma models and complex geometry that I find remarkable: To each superspace formulation of the sigma model, be it $N=(2,2)$, $N=(2,1)$, $N=(1,2)$ or $N=(1,1)$, there is always a natural corresponding formulation of the Generalized Kähler Geometry on the target space. I first introduce the relevant formulations of Generalized Kähler Geometry and then the sigma models. The results are collected from a number of papers where we have used sigma models as tools to probe the geometry: [1]-[14]. See also [15], [16] for related early discussions.

2. Formulations of Generalized Kähler Geometry

Generalized Kähler Geometry was defined by Gualtieri [18] in his PhD thesis on Generalized Complex Geometry. The latter subject was introduced by Hitchin in [19]. In [18] it is also described how GKG is a reformulation of the bihermitean geometry of [23], which we now turn to.

2.1 Generalized Kähler Geometry I; Bihermitean Geometry.

Bihermitean geometry is the set $(M,g,J_{(\pm)},H)$, i.e., a manifold M equipped with a metric g, two complex structures $J_{(\pm)}$ and a closed three-form H. The defining properties may be summarized as follows:

$$\boxed{\begin{array}{c} J^2_{(\pm)}=-\mathbb{1}\ , \quad J^t_{(\pm)}gJ_{(\pm)}=g\ , \quad \nabla^{(\pm)}J_{(\pm)}=0 \\[2ex] \Gamma^{(\pm)}=\Gamma^0\pm\tfrac{1}{2}g^{-1}H\ , \quad H=dB\ . \end{array}}$$

Table 1: Bihermitean 1

In words, the metric is hermitean with respect to <u>both</u> complex structures and these, in turn, are covariantly constant with respect to connections which are the sum of the Levi-Civita connection and a torsion formed from the closed three-form. Locally, the three-form may be expressed in terms of a potential two-form B. This B-field, or NSNS two-form, is conveniently combined with the metric into one tensor E:

$$E:=g+B\ . \tag{2.1}$$

A reformulation of the data in Table.1 more adapted to Generalized Complex Geometry is as the set $(M,g,J_{(\pm)})$ supplemented with (integrability)conditions acoording to

$$J^2_{(\pm)} = -1\!\!1 \ , \quad J^t_{(\pm)} g J_{(\pm)} = g \ , \quad \omega_{(\pm)} := g J_{(\pm)}$$

$$d^c_{(+)}\omega_{(+)} + d^c_{(-)}\omega_{(-)} = 0 \ , \quad dd^c_{(\pm)}\omega_{(\pm)} = 0 \ ,$$

$$H := d^c_{(+)}\omega_{(+)} = -d^c_{(-)}\omega_{(-)}$$

Table 2: Bihermitean 2

Here $\omega_\pm$ are the generalizations of the Kähler forms for the two complex structures, d^c is the differential which reads $i(\bar{\partial} - \partial)$ in local coordinates where the complex structure is diagonal, and we see that the three-form is defined in terms of the basic data.

2.2 Generalized Kähler Geometry II; Description on $T \oplus T^*$

Generalized Complex Geometry [19], [18], is formulated on the sum of the tangent and cotangent bundles $T \oplus T^*$ equipped with an endomorphism which is a (generalized) almost complex structure, i.e., a map

$$\mathscr{J} : T \oplus T^* \to T \oplus T^* \ : \ \mathscr{J}^2 = -1\!\!1 \ . \tag{2.2}$$

The further requirements that turn $\mathscr{J}$ into a generalized complex structure is first that it preserves the natural pairing on $T \oplus T^*$

$$\mathscr{J}^t I \mathscr{J} = I \ , \quad I := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{2.3}$$

where the matrix expression refers to the coordinate basis (∂_μ, dx^ν) in $T \oplus T^*$, and second the integrability condition

$$\pi_\mp [\pi_\pm X, \pi_\pm Y]_C = 0 \ , \quad X, Y \in T \oplus T^*. \tag{2.4}$$

Here C denotes the Courant bracket [21], which for $X = x + \xi, Y = y + \eta \in T \oplus T^*$ reads

$$[X,Y]_C := [x,y] + \mathscr{L}_x \eta - \mathscr{L}_y \xi - \frac{1}{2} d(\imath_x \eta - \imath_y \xi) \ , \tag{2.5}$$

with the Lie bracket, Lie derivative and contraction of forms with vectorfields appearing on the right hand side. Generalized Kähler Geometry [18] requires the existence of two commuting such Generalized Complex Structures, i.e.:

$$\mathscr{J}^2_{(1,2)} = -1\!\!1 \quad [\mathscr{J}_{(1)}, \mathscr{J}_{(2)}] = 0 \ , \quad \mathscr{J}^t_{(1,2)} I \mathscr{J}_{(1,2)} = I \ , \quad \mathscr{G} := -\mathscr{J}_{(1)} \mathscr{J}_{(2)} \ , \tag{2.6}$$

with both GCSs satisfying (2.4) and the last line defines an almost product structure $\mathscr{G}$:

$$\mathscr{G}^2 = 1\!\!1 \ . \tag{2.7}$$

When formulated in $T \oplus T^*$, Kähler geometry satisfies these condition, and so does bihermitean geometry. In fact the Gualtieri map [18] gives the precise relation[1] to the data in Table 2:

$$\mathscr{J}_{(1,2)} = \begin{pmatrix} 1 & 0 \\ & \\ B & 1 \end{pmatrix} \begin{pmatrix} J_{(+)} \pm J_{(-)} & -(\omega^{-1}_{(+)} \mp \omega^{-1}_{(-)}) \\ & \\ \omega_{(+)} \mp \omega_{(-)} & -(J^t_{(+)} \pm J^t_{(-)}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ & \\ -B & 1 \end{pmatrix} \tag{2.8}$$

2.3 Generalized Kähler Geometry III; Local Symplectic Description

Bihermitean geometry emphasizes the complex aspect of generalized Kähler Geometry. There is another formulation where the (local) symplectic structure is in focus.

Given the bi-complex manifold $(M, J_{(\pm)})$, there exists locally defined non-degenerate "symplectic" two-forms $\mathscr{F}_{(\pm)}$ such that $d\mathscr{F}_{(\pm)} = 0$ and [12]

$$\boxed{\mathscr{F}_{(\pm)}(v, J_{(\pm)}v) > 0 \ , \quad d(\mathscr{F}_{(+)}J_{(+)} - J^t_{(-)}\mathscr{F}_{(-)}) = 0 \ .}$$

Table 3: Conditions on $\mathscr{F}$

In the first condition v is an arbitrary contravariant vector field and the condition says that $\mathscr{F}_{\pm}$ tames the complex structures $J_{(\pm)}$. The bihermitean data is recovered from

$$\begin{aligned} \mathscr{F}_{(\pm)} &= \frac{1}{2} i (B^{(2,0)}_{(\pm)} - B^{(0,2)}_{(\pm)}) \mp \omega_{(\pm)} \\ \mathscr{F}_{(+)} &= -\frac{1}{2} E^t_{(+)} J_{(+)} \ , \quad \mathscr{F}_{(-)} = -\frac{1}{2} J^t_{(-)} E^t_{(-)} \end{aligned} \tag{2.9}$$

where, e.g., $B^{(2,0)}_{(\pm)}$ refers to the holomorphic property of B under $J_{(\pm)}$.

2.4 Summary

As we have seen, the geometric data representing Generalized Kähler Geometry may be packaged in various equivalent ways as, e.g., $(M, g, H, J_{(\pm)})$, as $(M, g, J_{(\pm)})$ or as $(M, \mathscr{F}_{(\pm)}, J_{(\pm)})$. In each case, there is a complete description in terms of a Generalized Kähler potential K [4][2]. Unlike the Kähler case, the expressions are non-linear in second derivatives of K. E.g., restricting attention to the situation $[J_{(+)}, J_{(-)}] \neq \emptyset$, the left complex structure is given by

$$J_{(+)} = \begin{pmatrix} J & 0 \\ & \\ (K_{LR})^{-1}[J, K_{LL}] & (K_{LR})^{-1} J K_{LR} \end{pmatrix} \ , \tag{2.10}$$

[1] The derivation from sigma models is given in [6].

[2] The description is complete away from irregular points of certain poisson structures

where we we introduced local coordinates $(\mathbb{X}^L, \mathbb{X}^R)$, $L := \ell, \bar{\ell}$, $R := r, \bar{r}$, and K_{LR} is shorthand for the matrix

$$K_{LR} := \begin{pmatrix} \frac{\partial^2 K}{\partial \mathbb{X}^\ell \partial \mathbb{X}^r} & \frac{\partial^2 K}{\partial \mathbb{X}^\ell \partial \mathbb{X}^{\bar{r}}} \\ \\ \frac{\partial^2 K}{\partial \mathbb{X}^\ell \partial \mathbb{X}^r} & \frac{\partial^2 K}{\partial \mathbb{X}^\ell \partial \mathbb{X}^{\bar{r}}} \end{pmatrix} . \tag{2.11}$$

The metric is

$$g = \Omega[J_{(+)}, J_{(-)}] , \tag{2.12}$$

and the local symplectic structures have potential one-forms $\lambda_{(\pm)}$. E.g.,

$$\mathscr{F}_{(+)} = d\lambda_{(+)} , \quad \lambda_{(+)\ell} = iK_R J(K_{LR})^{-1} K_{L\ell} , \ldots \tag{2.13}$$

The relations may be extended to the whole manifold in terms of gerbes [12].

3. Sigma Models

The $d = 2$, $N = (2,2)$ supersymmetry algebra of covariant derivatives is

$$\{\mathbb{D}_\pm, \bar{\mathbb{D}}_\pm\} = i\partial_{\pm\!\!\!\!\pm} \tag{3.1}$$

The covariant derivatives can be used to constrain superfields. We shall need chiral, twisted chiral and left and right semichiral superfields [17]:

$$\begin{aligned} \bar{\mathbb{D}}_\pm \phi^a &= 0 , \\ \bar{\mathbb{D}}_+ \chi^{a'} = \mathbb{D}_- \chi^{a'} &= 0 , \\ \bar{\mathbb{D}}_+ \mathbb{X}^\ell &= 0 , \\ \bar{\mathbb{D}}_- \mathbb{X}^r &= 0 , \end{aligned} \tag{3.2}$$

and their complex conjugate. The collective indexnotation is taken to be; $c := a\ , \bar{a}$, $t := a', \bar{a}'$, and, as before, $L := \ell, \bar{\ell}$, $R := r, \bar{r}$.

3.1 Superspace I

The $(2,2)$ formulation of the $(2,2)$ sigma model uses the generalized Kähler Potential K directly:

$$S = \int \mathbb{D}_+ \bar{\mathbb{D}}_+ \mathbb{D}_- \bar{\mathbb{D}}_- K(\phi^c, \chi^t, \mathbb{X}^L, \mathbb{X}^R) \tag{3.3}$$

Note that K has many roles: as a Lagrangian as in (3.3), as a potential for the geometry, (2.10), (2.11), as a "prepotential" for the local symplectic form $\mathscr{F}$, (2.13), and, as shown in [4], as a generating function for symplectomorphisms between coordinates where $J_{(+)}$ and coordinates where $J_{(-)}$ are canonical.

3.2 Superspace II

To discuss reduction of the action (3.3) to $(2,1)$ superspace [14], we restrict the potential to $K(\mathbb{X}^L,\mathbb{X}^R)$ to simplify the expressions.

The reduction entails representing the $(2,2)$ right derivative as a sum of $(2,1)$ derivative and a generator of supersymmetry:

$$\mathbb{D}_- =: D_- - iQ_- \ , \tag{3.4}$$

and defining the $(2,1)$ components of a $(2,2)$ superfield as

$$\mathbb{X}| =: X \ , \ \ Q_-\mathbb{X}^L| =: \Psi^L_- \ . \tag{3.5}$$

The action (3.3) then reduces as

$$S = \int \mathbb{D}_+\bar{\mathbb{D}}_+D_- \left(K_L\Psi^L_- + K_RJD_-X^R\right) \ . \tag{3.6}$$

Here Ψ is a Lagrange multiplier field enforcing $\bar{\mathbb{D}}_+K_\ell = 0$ and its c.c., which are the $(2,1)$ components of the $(2,2)$ $\mathbb{X}^\ell$ and $\mathbb{X}^{\bar\ell}$ equations. We solve this by going to $(2,2)$ coordinates $(\mathbb{X}^L,\mathbb{Y}_L)$ [4], [14], whose $(2,1)$ components will now both be chiral. The action then reads

$$S = i\int \mathbb{D}_+\bar{\mathbb{D}}_+D_-(\lambda_{(+)\alpha}D_-\varphi^\alpha + c.c.) \tag{3.7}$$

with $\varphi^\alpha \in (X^\ell,Y_\ell)$ and $\bar{\mathbb{D}}_+\varphi^\alpha = 0$. This is the standard form of a $(2,1)$ sigma model [22] but with the vector potential now identified (up to factors) with $\lambda_{(+)}$ in (2.13), ($\mathscr{F}_{(+)} = d\lambda_{(+)}$). Of the two complex structures $J_{(\pm)}$ only $J_{(+)}$ is now manifest. The complex structure $J_{(-)}$ instead appears in the non-manifest supersymmetry

$$\delta\varphi^\alpha = \bar{\mathbb{D}}_+(\varepsilon J_{(-)i}^{\ \alpha}D_-\phi^i) \ , \quad \{\phi^i\} = \{\varphi^\alpha,\bar\varphi^{\bar\alpha}\} \tag{3.8}$$

Similarily, reduction of (3.3) to $(1,2)$ yields a model in which $J_{(-)}$ is the remaining manifest complex structure. It is found from the $(2,1)$ model by the replacement $+\to-$, and $L\to R$.

3.3 Superspace III

We may reduce the action (3.3) to $(1,1)$ superspace directly or via the $(2,1)$ formulation. The resulting action now involves the metric and B-fields in the combination (2.1) as geometric objects:

$$S = \int D_+D_-\left(D_+XED_-X\right) \ , \tag{3.9}$$

where we have supressed the indices. Starting from $(2,1)$ superspace and the action (3.6), the reduction goes via

$$\mathbb{D}_+ =: D_+ - iQ_+ \ , \quad Q_+\mathbb{X}^R| =: \Psi^R_+ \ , \tag{3.10}$$

and both the auxiliary spinors Ψ^L_- and Ψ^R_+ have been eliminated[3]. Both complex structures are now non-manifest and arise in the extra supersymmetry transformations as explained in [23].

[3]Note that these spinors have the role of Lagrange multipliers in the $(2,1)$ and $(1,2)$ formulations, but the role of auxiliary fields with algebraic field equations in the $(1,1)$ formulation

3.4 Summary

The various sigma models have different formulations of Generalized Kähler Geometry manifest. Thus the $(2,2)$ sigma model is written directly in terms of the generalized Kähler potential. The $(2,1)$ or $(1,2)$ model involves the one form $\lambda_{(+)}$ or $\lambda_{(-)}$ respectively, which connects it to the local symplectic formulation. The $(1,1)$ sigma model, finally, is expressed directly in terms of the metric and B-field, making that aspect of the geometry manifest. These are also the objets that determine the $(0,0)$ formulation, i.e., the component formulation of the sigma model.

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References

[1] U. Lindstrom, *Generalized N = (2,2) supersymmetric nonlinear sigma models,* Phys. Lett. B **587**, 216 (2004) [hep-th/0401100].

[2] U. Lindstrom, R. Minasian, A. Tomasiello and M. Zabzine, *Generalized complex manifolds and supersymmetry,* Commun. Math. Phys. **257**, 235 (2005) [hep-th/0405085].

[3] U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, *Generalized Kahler geometry and manifest N = (2,2) supersymmetric nonlinear sigma-models,* JHEP **0507**, 067 (2005) [hep-th/0411186].

[4] U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, *Generalized Kahler manifolds and off-shell supersymmetry,* Commun. Math. Phys. **269**, 833 (2007) [hep-th/0512164].

[5] A. Bredthauer, U. Lindstrom and J. Persson, *First-order supersymmetric sigma models and target space geometry,* JHEP **0601**, 144 (2006) [hep-th/0508228].

[6] A. Bredthauer, U. Lindstrom, J. Persson and M. Zabzine, *Generalized Kahler geometry from supersymmetric sigma models,* Lett. Math. Phys. **77**, 291 (2006) [hep-th/0603130].

[7] U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, *Linearizing Generalized Kahler Geometry,* JHEP **0704**, 061 (2007) [hep-th/0702126].

[8] U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, *A potential for Generalized Kahler Geometry,* IRMA Lect. Math. Theor. Phys. **16**, 263 (2010) [hep-th/0703111].

[9] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge and M. Zabzine, *New N = (2,2) vector multiplets,* JHEP **0708**, 008 (2007) [arXiv:0705.3201 [hep-th]].

[10] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge and M. Zabzine, *T-duality and Generalized Kahler Geometry,* JHEP **0802**, 056 (2008) [arXiv:0707.1696 [hep-th]].

[11] U. Lindstrom, M. Rocek, I. Ryb, R. von Unge and M. Zabzine, *Nonabelian Generalized Gauge Multiplets,* JHEP **0902**, 020 (2009) [arXiv:0808.1535 [hep-th]].

[12] C. M. Hull, U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, *Generalized Kahler geometry and gerbes,* JHEP **0910**, 062 (2009) [arXiv:0811.3615 [hep-th]].

[13] C. M. Hull, U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, *Generalized Calabi-Yau metric and Generalized Monge-Ampere equation*, JHEP **1008**, 060 (2010) [arXiv:1005.5658 [hep-th]].

[14] C. Hull, U. Lindstrom, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler Geometry in (2,1) superspace*, arXiv:1202.5624 [hep-th].

[15] A. Sevrin and J. Troost, *Off-shell formulation of N=2 nonlinear sigma models*, Nucl. Phys. B **492**, 623 (1997) [hep-th/9610102].

[16] J. Bogaerts, A. Sevrin, S. van der Loo and S. Van Gils, *Properties of semichiral superfields*, Nucl. Phys. B **562**, 277 (1999) [hep-th/9905141].

[17] T. H. Buscher, *Quantum Corrections And Extended Supersymmetry In New Sigma Models*, Phys. Lett. B **159**, 127 (1985).

[18] M. Gualtieri, *Generalized complex geometry*, Oxford University DPhil thesis, [arXiv:math/0401221].

[19] N. Hitchin, *Generalized Calabi-Yau manifolds*, Quart. J. Math. Oxford Ser. **54**, 281-308 (2003). [math/0209099 [math-dg]].

[20] S. J. Gates, Jr., C. M. Hull, M. Rocek, *Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models*, Nucl. Phys. **B248**, 157 (1984).

[21] T. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc., 319:631-661, (1990)

[22] M. Abou Zeid and C. M. Hull, *The gauged (2,1) heterotic sigma model*, Nucl. Phys. B **513**, 490 (1998) [arXiv:hep-th/9708047].

[23] S. J. Gates, Jr., C. M. Hull, M. Rocek, *Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models*, Nucl. Phys. **B248**, 157 (1984).